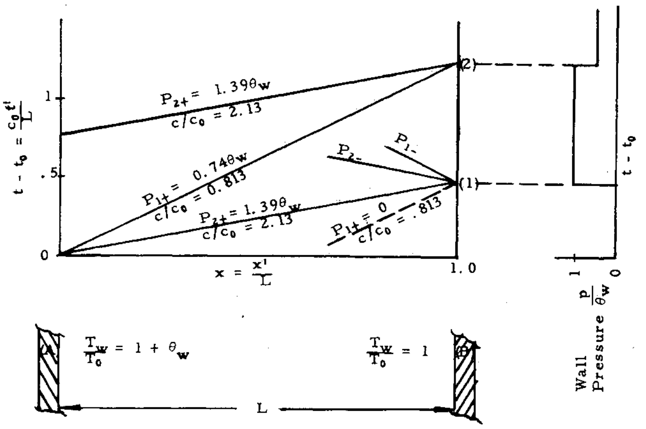

Fig. 1 Thermal disturbance propagation in very rarefied-gas field

Here p is the perturbation pressure, θ the perturbation temperature, u the velocity, τ the normal stress, and q the heat flux. The relaxation time is defined by $t_f = \mu_0/p_0$. In Eqs. (1) and (2), x,t, and the propagation velocities c_1 and c_2 are written in the dimensionless form

$$\frac{L}{x'} = \frac{1}{x} \qquad \frac{L}{c_0 t'} = \frac{1}{t} \qquad \frac{c_1}{c_0} = 0.813 \qquad \frac{c_2}{c_0} = 2.13$$

As an illustrative example, the disturbance produced by a plate suddenly heated in a rarefied-gas field, initially in equilibrium at a temperature T_0, can be calculated assuming $F = H = 0$, $L/c_0 \ll t_f$ and that specular reflection does not occur at the plate. The field particles are absorbed and reemitted with a Maxwellian distribution at the wall temperature T_w. Continuity of mass and the equation of state require that[2]

$$p = 0 = s_w + (\theta_w/2) \qquad p_w = \theta_w/2 \qquad u = 0$$

The characteristic values at the heated wall are $P_{2+} = 1.39\,\theta_w$, $P_{1+} = 0.74\theta_w$. At the unheated wall (1), $\theta = 0$, and the fast characteristic results in an average pressure $p = 1.08\theta_w$. At wall (2), the slow and fast characteristics yield an average pressure $p = 0.5\theta_w$ (Fig. 1).

The heated plate consequently produces a pressure disturbance that is transmitted by the rarefied-gas field and results in a positive pressure or repulsive force at (B) when $\theta_w > 0$ and a negative pressure or attractive force when $\theta_w < 0$.

This thermal disturbance propagation in the rarefied-gas field which occurs in the limit $L/t_f c_0 \ll 1$, $F = H = 0$ is unique in that relatively few field particle collisions occur during the propagation. Disturbances originated at a boundary will not be altered during propagation by other disturbances existing in the field and will be altered only by collisions at another boundary. Therefore, the thermal disturbances initiated at boundaries will propagate unchanged in the field.

In the limit $t_f \gg t$, when $F = H = 0$, the one-dimensional equations for longitudinal disturbances can be written in the form

$$\frac{\partial}{\partial t}\left(\theta - 0.51p - 0.11\frac{\tau}{p_0}\right) + 0.813\,\frac{\partial}{\partial x}\left(0.33\,\frac{q}{p_0 c_0} - 0.42\,\frac{u}{c_0}\right) = 0 \quad (5)$$

$$\frac{\partial}{\partial t}\left(0.33\,\frac{q}{p_0 c_0} - 0.42\,\frac{u}{c_0}\right) + 0.813\,\frac{\partial}{\partial x}\left(\theta - 0.51p - 0.11\,\frac{\tau}{p_0}\right) = 0 \quad (6)$$

$$\frac{\partial}{\partial t}\left(\theta + 0.78p + 1.18\,\frac{\tau}{p_0}\right) + 2.13\,\frac{\partial}{\partial x}\left(0.85\,\frac{q}{p_0 c_0} + 1.66\,\frac{u}{c_0}\right) = 0 \quad (7)$$

$$\frac{\partial}{\partial t}\left(0.85\,\frac{q}{p_0 c_0} + 1.66\,\frac{u}{c_0}\right) + 2.13\,\frac{\partial}{\partial x}\left(\theta + 0.78p + 1.18\,\frac{\tau}{p_0}\right) = 0 \quad (8)$$

By elimination of the terms containing u and q, the following propagation equations are obtained for the longitudinal temperature, stress, and pressure disturbances:

$$\frac{\partial^2}{\partial t^2}\left(\theta - 0.51p - 0.11\,\frac{\tau}{p_0}\right) = (0.813)^2\,\frac{\partial^2}{\partial x^2}\left(\theta - 0.51p - 0.11\,\frac{\tau}{p_0}\right) \quad (9)$$

$$\frac{\partial^2}{\partial t^2}\left(\theta + 0.78p + 1.18\,\frac{\tau}{p_0}\right) = (2.13)^2\,\frac{\partial^2}{\partial x^2}\left(\theta + 0.78p + 1.18\,\frac{\tau}{p_0}\right) \quad (10)$$

The equations for the propagation of small plane disturbances, Eqs. (1) and (2), also may be written in the following form:

$$[\mathbf{n}(1/c)(\partial/\partial t) \pm \nabla]\cdot\mathbf{P}_{1,2\pm} = 0$$

$$[\mathbf{n}(1/c)(\partial/\partial t) \pm \nabla]\;\mathbf{x}\;\mathbf{P}_{1,2\pm} = 0$$

where, $t_f \gg t$, $F = H = 0$, $\mathbf{P}_{1,2} = \mathbf{i}P_{1,2}$, c is the dimensionless propagation velocity, and $\mathbf{n}$ is a unit vector along the direction of propagation. The forward propagating plane longitudinal disturbances consequently satisfy equations of the form

$$\nabla\cdot\mathbf{P} + (1/c)(\partial\mathbf{n}\cdot\mathbf{P})/\partial t] = 0 \quad (11)$$

$$\nabla\;\mathbf{x}\;\mathbf{P} = 0 \quad (12)$$

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A Further Note on Propagation of Transverse Disturbances in Rarefied-Gas Flows

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Lees and Yang[1] recently have shown that the two-dimensional Grad equations for the rarefied-gas field, when applied to the Rayleigh problem, indicate the propagation of small transverse shear disturbances along distinct char-

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acteristics with a velocity[1,2] $c = \pm(\frac{7}{5}p_0/\rho_0)^{1/2}$. The propagation phenomena occur in the limit that the flow time, the time during which the propagation occurs, is small compared to the relaxation time, the mean time between field particle collisions. Similar propagation phenomena were shown to exist in the analysis of Lees.[3,4]

The existence of this small-disturbance propagation phenomenon along distinct characteristics suggests that, in this limit, the linearized small-disturbance propagation equations for the Rayleigh problem also satisfy equations of the form

$$[(\partial/\partial t) \pm c(\partial/\partial y)]\, Q(y, t) = 0 \qquad (1)$$

when no external sources exist in the field. $Q(y, t)$ is defined by $Q(y, t) = \alpha P_{xy}(y, t) + \beta u(y, t) + \gamma q_x(y, t)$, where α, β, and γ are constants, P_{xy} is the shear stress, u the shear velocity, and q_x the heat flux.

With the assumption of small impulsive velocities and small temperature differences, the system of partial differential equations for the Rayleigh problem may be written in the following form:

$$\frac{\partial}{\partial t}\left(\frac{c_1}{c_0}\frac{u}{c_1}\right) + c_1\frac{\partial}{\partial y}\left(\frac{c_0}{c_1}\frac{P_{xy}}{p_0}\right) = 0 \qquad (2)$$

$$\frac{\partial}{\partial t}\left(\frac{P_{xy}}{p_0}\right) + c_1\frac{\partial}{\partial y}\left(\frac{u}{c_1}\right) + c_1\frac{\partial}{\partial y}\left(\frac{2}{5}\frac{q_x}{p_0c_1}\right) = -\frac{1}{t_f}\frac{P_{xy}}{p_0} \qquad (3)$$

$$\frac{\partial}{\partial t}\frac{c_1}{c_0}\frac{q_x}{p_0c_1} + c_1\frac{\partial}{\partial y}\left(\frac{c_0}{c_1}\frac{P_{xy}}{p_0}\right) = -\frac{2}{3}\frac{1}{t_f}\frac{q_x}{p_0c_0} \qquad (4)$$

$$\frac{\partial p}{\partial t} - \frac{\partial \theta}{\partial t} + c_2\frac{\partial}{\partial y}\left(\frac{v}{c_2}\right) = 0 \qquad (5)$$

$$\frac{\partial}{\partial t}\left(\frac{c_2 v}{c_0 c_2}\right) + c_2\frac{\partial}{\partial y}\left(\frac{c_0}{c_2}p\right) + c_2\frac{\partial}{\partial y}\left(\frac{c_0}{c_2}\frac{P_{yy}}{p_0}\right) = 0 \qquad (6)$$

$$\frac{\partial}{\partial t}(p) + c_2\frac{\partial}{\partial y}\left(\frac{5}{3}\frac{v}{c_2}\right) + c_2\frac{\partial}{\partial y}\left(\frac{2}{3}\frac{q_y}{p_0c_2}\right) = 0 \qquad (7)$$

$$\frac{\partial}{\partial t}\left(\frac{P_{yy}}{p_0}\right) + c_2\frac{\partial}{\partial y}\left(\frac{4}{3}\frac{v}{c_2}\right) + c_2\frac{\partial}{\partial y}\left(\frac{8}{15}\frac{q_y}{p_0c_2}\right) = -\frac{1}{t_f}\frac{P_{yy}}{p_0} \qquad (8)$$

$$\frac{\partial}{\partial t}\left(\frac{c_2}{c_0}\frac{q_y}{p_0c_2}\right) + c_2\frac{\partial}{\partial y}\left(\frac{5}{2}\frac{c_0\theta}{c_2}\right) + c_2\frac{\partial}{\partial y}\left(\frac{c_0}{c_2}\frac{P_{yy}}{p_0}\right) = -\frac{2}{3}\frac{1}{t_f}\frac{q_y}{p_0c_0} \qquad (9)$$

$$\frac{\partial}{\partial t}\left(\frac{P_{xx}}{p_0}\right) - c_2\frac{\partial}{\partial y}\left(\frac{2}{3}\frac{v}{c_2}\right) - c_2\frac{\partial}{\partial y}\left(\frac{4}{15}\frac{q_y}{c_2p_0}\right) = -\frac{1}{t_f}\frac{P_{xx}}{p_0} \qquad (10)$$

and

$$p = s + \theta \qquad c_0^2 = p_0/\rho_0 = RT_0$$

where $u' = u$, $v' = v$, $\rho'/\rho_0 = 1 + s$, $p'/p_0 = 1 + p$, $T'/T_0 = 1 + \theta$, $P_{xx}' = P_{xx}$, $P_{xy}' = P_{xy}$, $P_{yy}' = P_{yy}$, $q_x' = q_x$, $q_y' = q_y$, $t_f = \mu_0/p_0$. The unprimed quantities denote small perturbations in the undisturbed quantities denoted here by primes. Here ρ' is the gas density, T' the temperature, u' the shear velocity, v' the normal velocity, p' the pressure, P_{ij}' the stress tensor, q_i' the heat flux, c_1 and c_2 are propagation velocities, and a relaxation time t_f is defined by μ_0/p_0. Equations (2–4) describing the propagation of the transverse components may be separated from the equations describing the propagation of the longitudinal components, i.e., the transverse and longitudinal propagation phenomena are not coupled in the field. On multiplying Eqs. (3) and (4) by α and β, respectively, the following compatibility conditions

$$\frac{c_1}{c_0} = \alpha \qquad \alpha = \frac{c_0}{c_1} + \frac{c_0}{c_1}\beta \qquad \beta\frac{c_1}{c_0} = \frac{2}{5}\alpha$$

are obtained for the existence of solutions of the form (1). The solution yields the propagation velocity $c_1/c_0 = \pm(\frac{7}{5})^{1/2}$ as previously obtained by Grad[2] and Lees and Yang[1] and the following propagation equations:

$$\left[\frac{\partial}{\partial t'} \pm \left(\frac{7}{5}\right)^{1/2} c_0\frac{\partial}{\partial y'}\right]\left[\frac{P_{xy}}{p_0} \pm \left(\frac{u}{c_1} + \frac{2}{5}\frac{q_x}{p_0c_1}\right)\right] = -\frac{1}{t_f}\left(\frac{P_{xy}}{p_0} \pm \frac{4}{3(35)^{1/2}}\frac{q_x}{p_0c_0}\right) \qquad (11)$$

where $c_1 = |\pm c_1|$. On multiplying Eq. (11) by L/c_0, the characteristic equations may be written in the nondimensional form

$$\left[\frac{\partial}{\partial t} \pm \left(\frac{7}{5}\right)^{1/2}\frac{\partial}{\partial y}\right]\left[\frac{P_{xy}}{p_0} \pm \left(\frac{u}{c_1} + \frac{2}{5}\frac{q_x}{p_0c_1}\right)\right] = -\frac{L}{t_fc_0}\left(\frac{P_{xy}}{p_0} \pm \frac{4}{3(35)^{1/2}}\frac{q_x}{p_0c_0}\right) \qquad (11')$$

where $(c_0/y')(L/c_0)$ is replaced by $1/y$ and $(1/t')(L/c_0)$ is replaced by $1/t$.

When $L/c_0 \ll t_f$, Eqs. (2–4) also may be written in the form

$$\frac{\partial}{\partial t}\left(\frac{P_{xy}}{p_0}\right) + c_1\frac{\partial}{\partial y}\left(\frac{u}{c_1} + \frac{2}{5}\frac{q_x}{p_0c_1}\right) = 0$$

$$\frac{\partial}{\partial t}\left(\frac{u}{c_1} + \frac{2}{5}\frac{q_x}{p_0c_1}\right) + c_1\frac{\partial}{\partial y}\left(\frac{P_{xy}}{p_0}\right) = 0$$

which yield the propagation equations

$$\frac{\partial^2}{\partial t^2}\left(\frac{P_{xy}}{p_0}\right) = c_1^2\frac{\partial^2}{\partial y^2}\left(\frac{P_{xy}}{p_0}\right) \qquad (12)$$

$$\frac{\partial^2}{\partial t^2}\left(\frac{u}{c_1} + \frac{2}{5}\frac{q_x}{p_0c_1}\right) = c_1^2\frac{\partial^2}{\partial y^2}\left(\frac{u}{c_1} + \frac{2}{5}\frac{q_x}{p_0c_1}\right) \qquad (13)$$

Equations (5–9) describing the propagation of small longitudinal disturbances are identical with the one-dimensional equations for the propagation of small longitudinal disturbances. These equations previously were shown[5] to possess the characteristic propagation velocities $c_2/c_0 = \pm 0.813$ and ± 2.13 and the characteristic solutions

$$\{(\partial/\partial t) \pm 0.813\, c_0(\partial/\partial y)\}P_{1\pm} = 0 \qquad (14)$$

$$\{(\partial/\partial t) \pm 2.13\, c_0(\partial/\partial y)\}\, P_{2\pm} = 0 \qquad (15)$$

where

$$P_{1\pm} = \left(\theta - 0.51p - 0.11\frac{P_{yy}}{p_0}\right) \pm \frac{1}{c_0}\left(0.33\frac{q_y}{p_0} - 0.42v\right)$$

$$P_{2\pm} = \left(\theta + 0.78\, p + 1.18\frac{P_{yy}}{p_0}\right) \pm \frac{1}{c_0}\left(0.85\frac{q_y}{p_0} + 1.66v\right)$$

Assuming the existence of external heat addition $H(y, t)$ and an external force $F(y, t)$, the equations become

$$\left\{\frac{\partial}{\partial t} \pm 0.813\frac{\partial}{\partial y}\right\}P_{1\pm} = \left(0.487\frac{H}{p_0} \mp 0.417\frac{F}{c_0}\right)\frac{L}{c_0} + \frac{L}{t_fc_0}\left(0.15\frac{P_{yy}}{p_0} \mp 0.4\frac{q_y}{p_0c_0}\right) \qquad (14')$$

$$\left\{\frac{\partial}{\partial t} \pm 2.13\frac{\partial}{\partial y}\right\}P_{2\pm} = \left(1.78\frac{H}{p_0} \pm 1.66\frac{F}{c_0}\right)\frac{L}{c_0} - \frac{L}{t_fc_0}\left(1.57\frac{P_{yy}}{p_0} \pm 0.4\frac{q_y}{p_0c_0}\right) \qquad (15')$$

Equations (8) and (10) may be combined to yield an equation relating the normal stresses which, for $L/c_0 \ll t_f$, becomes

$$(\partial/\partial t)(P_{xx}/p_0) = -\tfrac{1}{2}(\partial/\partial t)(P_{yy}/p_0) \qquad (16)$$

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Unsymmetrical Buckling of Shallow Spherical Shells

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THE numerical results for the critical pressure for unsymmetrical buckling of clamped, shallow, spherical shells recently were presented by Weinitschke.[1] The author of the present note independently has obtained results for the same problem which are in striking disagreement with those of Weinitschke. The governing differential equations used agree with those used by Weinitschke. The buckling pressures were calculated numerically. The final results are shown in Fig. 1 together with some available experimental data.[2, 3] The pressure parameter p is defined as the ratio of the external pressure q to the classical buckling pressure q_0 of the complete spherical shell of the same radius of curvature and thickness; n is the number of waves along the circumferential direction appearing in the buckling mode. The disagreement with Weinitschke's results is displayed in Fig. 2, but the reason for this disagreement remains unknown.

According to the author's results, unsymmetrical buckling $(\mathbf{n} \neq \mathbf{0})$ occurs only for $\lambda > 5.5$. For $\lambda < 5.5$, buckling occurs

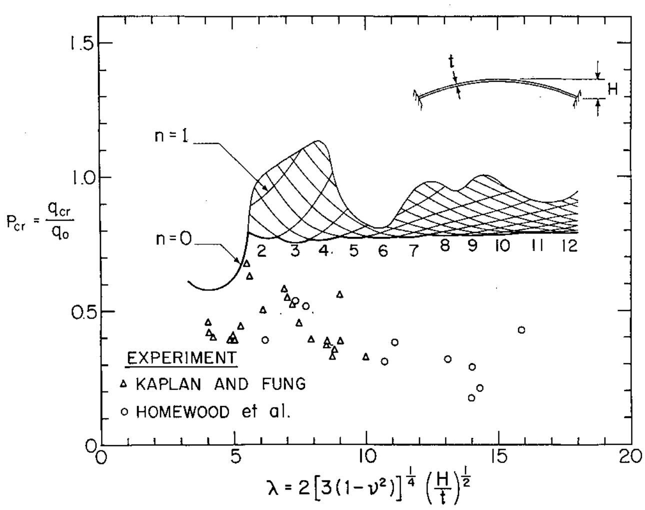

Fig. 1 Calculated buckling pressures of clamped, shallow, spherical shells and experimental results

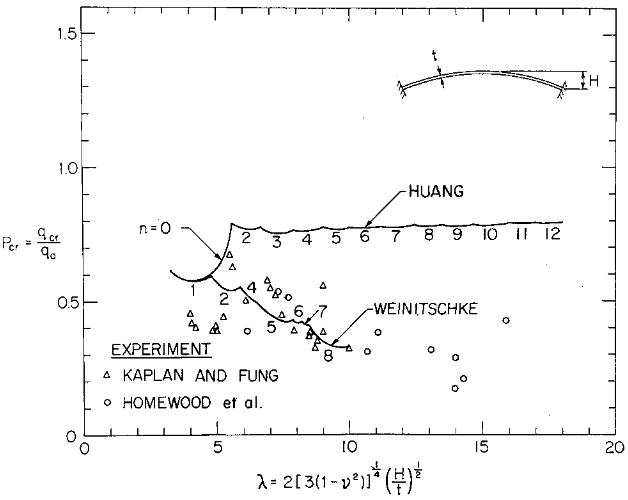

Fig. 2 Calculated buckling pressures vs Weinitschke's results

by axisymmetrical snapping. As λ keeps increasing, the buckling mode shows more and more waves along the circumferential direction and also shows a distinct boundary layer near the edge of the shell along the radial direction when λ is high. An asymptotic value of the buckling pressure is found to be 0.864 when λ approaches infinity and the ratio of n/λ approaches 0.817. Initial imperfections of the shell are presumed to be the source of the discrepancy between theory and experiment.

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On Axially Symmetric, Turbulent, Compressible Mixing in the Presence of Initial Boundary Layer

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RECENT experimental results[1] have shown that the mixing of heterogeneous gases having an initial velocity ratio close to unity occurs faster than is predicted by classical eddy-viscosity theory. The theoretical analysis of two uniform streams of different gases but of nearly equal velocity, performed with the usual assumptions for eddy viscosity and Prandtl number equal to a constant,[2] shows that mixing will take place very slowly, i.e., at the rate corresponding to laminar diffusion. It has been suggested that the difference

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